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# Approximate Solutions of the Dirac Equation for the Manning-Rosen Potential Including the Spin-Orbit Coupling Term

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**Abstract** The bound-state solutions of the Dirac equation for the Manning-Rosen potential are presented approximately for arbitrary spin-orbit quantum number $\kappa$. The energy eigenvalues and corresponding two-component spinors of the two Dirac particles are obtained in the closed form by using the framework of the spin symmetry and pseudospin symmetry concept. Two special cases $\kappa = \pm 1$ and the Hulthén potential are briefly investigated.

**Keywords** Dirac equation · Spin symmetry · Pseudospin symmetry · Manning-Rosen potential · Hulthén potential

## 1 Introduction

The pseudospin symmetry with the nuclear shell model was established over thirty years ago [1, 2]. After that, pseudospin symmetry has been applied to many systems in nuclear physics and related areas [3, 4], such as the deformed nuclei [5] and the super-deformation [6]. Ginocchio [3, 4] has shown that the pseudospin symmetry arises from the near equality of the magnitude of the attractive scalar potential $S(r)$ and repulsive vector potential $V(r)$, i.e., $S(r) \sim -V(r)$ in nuclei. The pseudospin symmetry refers to a quasi-degeneracy of the single particle doublets and can be characterized with the quantum numbers $(n, l, j = l + 1/2)$ and $(n-1, l+2, j = l+3/2)$, where $n$, $l$ and $j$ are the radial, orbital and total angular momentum quantum numbers for a single particle, respectively. The total angular momentum is given as $j = \tilde{l} + \tilde{s}$, where $\tilde{l} = l + 1$ is a pseudo-angular momentum and $\tilde{s} = 1/2$ is a pseudospin angular momentum. The spin symmetry and pseudospin symmetry occur for $\Delta(r) = \text{const}$ and $\Sigma(r) = \text{const}$ for the Dirac equation [7]. Some authors have applied pseudospin symmetry on several physical potentials, such as the harmonic oscillator [8–11], the Wood-Saxon potential [12], the Morse potential [13, 14], the Hulthén potential [15], the Eckart potential [16–18], the three-parameter potential function as a diatomic molecule model [19] and the Pöschl-Teller potential [20].

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The Manning-Rosen potential [21, 22] is an important diatomic molecular potential model. In chemistry, the Manning-Rosen potential can be used as the providing better fits to the Rydberg-Klein-Ress curves for different diatomic molecules [23, 24]. We consider reduces mass for diatomic molecular potential model. When the nuclei have masses $m_1$ and $m_2$, the reduces mass is defined as $\mu = m_1 m_2/(m_1 + m_2)$ and in this point the diatomic molecular model can be included to the spin symmetry and pseudospin symmetry concept. The Manning-Rosen potential is also special case of the Tietz potential [25]. In [26], the authors analyzed the bound-state solutions of the s-wave Klein-Gordon equation and Dirac equation with equal scalar and vector Manning-Rosen potential. However, as far as we know, one has not reported the investigation of the spin symmetry and pseudospin symmetry solution of the Dirac equation for the Manning-Rosen potential with spin-orbit coupling term.

Motivated by the success made by some authors in finding approximate analytical solution of the Dirac equation with spin-orbit coupling term [12–18], we solve approximately the Dirac equation with the Manning-Rosen potential for the spin-orbit quantum number $\kappa$. We obtained the energy eigenvalues under consideration of spin symmetry and pseudospin symmetry case. The corresponding upper and lower spinor components are expressed by the generalized hypergeometric functions. This paper is organized as follows. In Sect. 2, approximate closed forms of the energy eigenvalues and corresponding two Dirac spinors are obtained by using the framework of the spin symmetry and pseudospin symmetry concept. In Sect. 3, two special cases $l = \tilde{l} = 0$ and the Hulthén potential are briefly studied. The concluding remark are given in Sect. 4.

## 2 Analytical Solution of the Dirac-Manning-Rosen Problem

The Dirac equation in units $\hbar = c = 1$ with both a scalar potential $S(r)$ and a vector potential $V(r)$ is

$$\left[\boldsymbol{\alpha}.\mathbf{p} + \beta\,(M + S(r))\right]\Psi(r) = [E - V(r)]\,\Psi(r) \tag{1}$$

where $E$ is the relativistic energy, $M$ is the mass of a single particle, $\mathbf{p}$ is the momentum operator, $\boldsymbol{\alpha}$ and $\beta$ are the $4 \times 4$ Dirac matrices

$$\mathbf{p} = -i\nabla, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ \boldsymbol{\sigma}_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

where $I$ is the unit matrix and $\boldsymbol{\sigma}_i$ is the Pauli matrices, respectively. For a particle in a central field, the total angular momentum $\mathbf{J}$ and $\hat{K} = -\beta(\boldsymbol{\alpha}.\mathbf{L} + 1)$ commute with the Dirac Hamiltonian where $\mathbf{L}$ is the orbital angular momentum. For a given total angular momentum $j$, the eigenvalues of the $\hat{K}$ are $\kappa = \pm(j + 1/2)$ where $-$ for aligned spin and $+$ for unaligned spin. The wavefunctions can be classified according to their angular momentum $j$ and spin-orbit quantum number $\kappa$ as follows

$$\Psi_{n\kappa}(r,\theta,\phi) = \frac{1}{r}\begin{bmatrix} F_{n\kappa}(r)Y^{l}_{jm}(\theta,\phi) \\ iG_{n\kappa}(r)Y^{\tilde{l}}_{jm}(\theta,\phi) \end{bmatrix} \tag{2}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are upper and lower components, $Y^{l}_{jm}(\theta,\phi)$ and $Y^{\tilde{l}}_{jm}(\theta,\phi)$ are the spherical harmonic functions. $n$ is the radial quantum number and $m$ is the projection of the angular momentum on the $z$ axis. The orbital angular momentum quantum numbers $l$ and

$\tilde{l}$ represent to the spin and pseudospin quantum numbers. For a given spin-orbit quantum number $\kappa = \pm 1, \pm 2, \ldots$, the orbital angular momentum and pseudo-orbital angular momentum are given by $l = |\kappa + 1/2| - 1/2$ and $\tilde{l} = |\kappa - 1/2| - 1/2$. Substituting (2) into (1), we obtain two couple equations for the radial part of the Dirac equation as follows

$$\left(\frac{d}{dr} + \frac{\kappa}{r}\right) F_{n\kappa}(r) = [M + E_{nk} - V(r) + S(r)]\, G_{n\kappa}(r), \tag{3a}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r}\right) G_{n\kappa}(r) = [M - E_{nk} + V(r) + S(r)]\, F_{n\kappa}(r). \tag{3b}$$

Eliminating $G_{n\kappa}(r)$ into (3a) and $F_{n\kappa}(r)$ into (3b), we get two second-order differential equations for the upper and lower spinor components as follows

$$\left(\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) + \frac{\frac{d\Delta}{dr}(\frac{d}{dr} + \frac{\kappa}{r})}{M + E_{n\kappa} - \Delta(r)}\right) F_{n\kappa}(r) = 0, \tag{4a}$$

$$\left(\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} - (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) - \frac{\frac{d\Sigma}{dr}(\frac{d}{dr} - \frac{\kappa}{r})}{M - E_{n\kappa} + \Sigma(r)}\right) G_{n\kappa}(r) = 0 \tag{4b}$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$.

### 2.1 Spin Symmetric Solution of the Manning-Rosen Potentail

In the case of exact spin symmetry, ($\frac{d\Delta(r)}{dr} = 0$, i.e., $\Delta(r) = C = \text{constant}$), (4a) becomes

$$\left(\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - (M + E_{n\kappa} - C)(M - E_{n\kappa} + \Sigma(r))\right) F_{n\kappa}(r) = 0 \tag{5}$$

where $\kappa = l$ for $\kappa < 0$ and $\kappa = -(l+1)$ for $\kappa > 0$. The energy eigenvalues depend on $n$ and $\kappa$, i.e., $E_{n\kappa} = E(n, \kappa(\kappa+1))$. It is well known as the spin symmetry [4]. In (5), we assume that $\Sigma(r)$ is the Manning-Rosen potential [21, 22], which is defined as

$$V(r) = \frac{1}{\Lambda b^2}\left[\frac{\alpha(\alpha-1)e^{-2r/b}}{(1 - e^{-r/b})^2} - \frac{Ae^{-r/b}}{1 - e^{-r/b}}\right], \quad \Lambda = 8\pi^2\mu \tag{6}$$

where $A$ and $\alpha$ are two dimensionless parameter but $b$ is the range of the potential and has dimension of length [27]. It can be shown that the Manning-Rosen potential remains invariant by mapping $\alpha \leftrightarrow 1-\alpha$ and has a minimum value $V(r_0) = \frac{A^2}{4b^2\alpha(1-\alpha)}$ at $r_0 = b\ln[1 + \frac{2\alpha(\alpha-1)}{A}]$ for $\alpha > 1$ to be obtained from first derivative of the potential respect to $r$. Moreover, when we chose $\alpha = 0$ or 1, the Manning-Rosen potential reduces to the Hulthén potential.

Substituting $\Sigma(r)$ into (5), defining

$$\lambda = b\sqrt{M^2 - E_{n\kappa}^2 - C(M - E_{n\kappa})},$$
$$\mathcal{A} = \frac{A}{\Lambda}(M + E_{n\kappa} - C), \tag{7}$$
$$\xi_+ = \frac{\alpha(\alpha-1)}{\Lambda}(M + E_{n\kappa} - C),$$

and using this definition, the upper spinor component becomes

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{1}{b^2}\left(-\lambda^2 - \frac{\xi_+ e^{-2r/b}}{(1-e^{-r/b})^2} + \frac{\mathcal{A}e^{-r/b}}{1-e^{-r/b}}\right)\right]F_{n\kappa}(r) = 0. \tag{8}$$

The Dirac equation for the Manning-Rosen potential given by (8) is analytically solvable only for $\kappa = -1$ ($l = 0$) case. To obtain analytical approximate solutions for the Manning-Rosen type potential with spin-orbit coupling term, we have to use an approximation. In order to deal with the spin-orbit coupling term, some authors [12–18] have used the following approximation

$$\frac{1}{r^2} \approx \frac{1}{b^2}\frac{e^{-r/b}}{(1-e^{-r/b})^2} \tag{9}$$

which is only valid for large values of the parameter $b$. Defining a new variable of the form $z = e^{-r/b}$ and using (9) into (8), we obtain

$$z^2\frac{d^2}{dz^2}F_{n\kappa}(z) + z\frac{d}{dz}F_{n\kappa}(z) - \left[\lambda^2 + \frac{\kappa(\kappa+1)z}{(1-z)^2} + \frac{\xi_+ z^2}{(1-z)^2} - \frac{\mathcal{A}z}{1-z}\right]F_{n\kappa}(z) = 0. \tag{10}$$

The upper spinor component $F_{n\kappa}(z)$ has to satisfy the boundary conditions, i.e., $F_{n\kappa}(0) = 0$ at $z \to 0$ ($r \to \infty$) and $F_{n\kappa}(1) = 0$ at $z \to 1$ ($r \to 0$). As a result, we may take the upper spinor component $F_{n\kappa}(z)$ of the forms

$$F_{n\kappa}(z) = (1-z)^{1+\delta} z^{\lambda} f_{n\kappa}(z) \tag{11}$$

where

$$\delta = \frac{1}{2}\left(-1 + \sqrt{(1+2\kappa)^2 + 4\xi_+}\right). \tag{12}$$

We remark that $\delta$ is still invariant by mapping $\alpha \leftrightarrow 1-\alpha$. Substituting (11) into (10), we have second-order differential equation as follows

$$(1-z)z\frac{d^2}{dz^2}f_{n\kappa}(z) + [2\lambda + 1 - (2\lambda + 2\delta + 3)z]\frac{d}{dz}f_{n\kappa}(z)$$
$$+ [\mathcal{A} - \kappa(\kappa+1) - (\delta+1)(1+2\lambda)]f_{n\kappa}(z) = 0. \tag{13}$$

The solution of this equation can be expressed in terms of the hypergeometric function, i.e.,

$$f_{n\kappa}(z) = {}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k!} \tag{14}$$

where

$$a = 1 + \delta + \lambda - \chi, \qquad b = 1 + \delta + \lambda + \chi, \tag{15a}$$

$$c = 1 + 2\lambda, \qquad \chi = \sqrt{\lambda^2 + \mathcal{A} + (\delta - \kappa)(1 + \delta + \kappa)}. \tag{15b}$$

When either $a$ or $b$ equals to a negative integer $-n$, the hypergeometric function $f_{n\kappa}(z)$ can be reduced to polynomial degree $n$. This shows that the hypergeometric function given by (14) can be finite under the following quantum condition:

$$a = 1 + \delta + \lambda - \chi = -n, \quad n = 0, 1, 2, 3, \dots. \tag{16}$$

Substituting this quantum condition into (15a), we have

$$\lambda = -\frac{(n+1)^2 - \mathcal{A} + \kappa(\kappa + 1) + (2n+1)\delta}{2(n + \delta + 1)}. \tag{17}$$

Using (7) and (12) into (17), we have an explicit expression for the energy eigenvalues of the Dirac equation with the Manning-Rosen potential

$$M^2 - E_{n\kappa}^2 - C(M - E_{n\kappa}) = \frac{1}{b^2}\left[\frac{(n+1)^2 - \mathcal{A} + \kappa(\kappa + 1) + (2n+1)\delta}{2(n + \delta + 1)}\right]^2. \tag{18}$$

This eigenvalue equation is still invariant under $\alpha \leftrightarrow 1 - \alpha$. It can be seen from (18) that the spin symmetric limit leads to quadratic energy eigenvalues. Therefore, the solution of this equation consists of positive and negative energy eigenvalues for each $n$ and $\kappa$. Ginocchio [7] has shown that there are only positive energy eigenvalues and no bound negative energy eigenvalues exist in the spin limit. In other words, only positive energy eigenvalues are chosen for the spin symmetric limit.

By using (15) we can write the lower spinor component

$$F_{n\kappa}(z) = N(1 - z)^{1+\delta} z^{\lambda} {}_2F_1(-n, n + 2 + 2(\delta + \lambda), 2\lambda + 1; z) \tag{19}$$

where $N$ is normalization constant.

### 2.2 Pseudospin Symmetric Solution of the Manning-Rosen Potentail

In the case of pseudospin symmetry ($\frac{d\Sigma}{dr} = 0$, i.e., $\Sigma(r) = C = \text{constant}$), (4b) becomes

$$\left(\frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} - (M - E_{n\kappa} + C)(M + E_{n\kappa} - \Delta(r))\right) G_{n\kappa}(r) = 0 \tag{20}$$

where $\kappa = -\tilde{l}$ for $\kappa < 0$ and $\kappa = \tilde{l} + 1$ for $\kappa > 0$. The energy eigenvalues depend on $n$ and $\kappa$, i.e., $E_{n\kappa} = E(n, \kappa(\kappa - 1))$. According to the pseudospin symmetry, the state with $j = \tilde{l} \pm 1/2$ are degenerate for $\tilde{l} \neq 0$ [4]. Equation (20) can not be solved exactly for the Manning-Rosen potential for $\kappa \neq 1$ ($\tilde{l} = 0$) by using the standard method. Hence, we have to use an approximation for the spin-orbit coupling term as in the previous section. Therefore, substituting (6) into (20), using a new variable of the form $z = e^{-r/b}$ and defining

$$\lambda = b\sqrt{M^2 - E_{n\kappa}^2 + C(M + E_{n\kappa})},$$
$$\overline{\mathcal{A}} = \frac{A}{\Lambda}(M - E_{n\kappa} + C), \tag{21}$$
$$\xi_- = \frac{\alpha(\alpha-1)}{\Lambda}(M - E_{n\kappa} + C),$$

the lower spinor component becomes

$$z^2\frac{d^2}{dz^2}G_{n\kappa}(z) + z\frac{d}{dz}G_{n\kappa}(z) - \left[\lambda^2 + \frac{\kappa(\kappa-1)z}{(1-z)^2} - \frac{\xi_- z^2}{(1-z)^2} + \frac{\overline{\mathcal{A}}z}{1-z}\right]G_{n\kappa}(z) = 0. \tag{22}$$

The lower spinor component $G_{n\kappa}(z)$ has to satisfy the boundary conditions, i.e., $G_{n\kappa}(0) = 0$ at $z \to 0$ $(r \to \infty)$ and $G_{n\kappa}(1) = 0$ at $z \to 1$ $(r \to 0)$. As a result, we may take the lower spinor component $G_{n\kappa}(z)$ of the forms

$$G_{n\kappa}(z) = (1-z)^{1+\delta} z^{\lambda} g_{n\kappa}(z) \tag{23}$$

where

$$\delta = \frac{1}{2}\left(-1 + \sqrt{(1-2\kappa)^2 - 4\xi_-}\right). \tag{24}$$

Equation (24) is still invariant by mapping $\alpha \leftrightarrow 1 - \alpha$. Substituting (23) into (22), we have second-order differential equation as follows

$$(1-z)z\frac{d^2}{dz^2}g_{n\kappa}(z) + [2\lambda + 1 - (2\lambda + 2\delta + 3)z]\frac{d}{dz}g_{n\kappa}(z)$$
$$- [\overline{\mathcal{A}} + \kappa(\kappa-1) + (\delta+1)(1+2\lambda)]g_{n\kappa}(z) = 0. \tag{25}$$

The solution of this equation can be expressed in terms of the hypergeometric function, i.e.,

$$g_{n\kappa}(z) = {}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k!} \tag{26}$$

where

$$a = 1 + \delta + \lambda - \chi, \qquad b = 1 + \delta + \lambda + \chi, \tag{27a}$$
$$c = 1 + 2\lambda, \qquad \chi = \sqrt{\lambda^2 - \overline{\mathcal{A}} + (\delta+\kappa)(1+\delta-\kappa)}. \tag{27b}$$

From the properties of the hypergeometric functions, the series given by (26) approaches infinity unless either $a$ or $b$ equals to a negative integer. Therefore, the lower Dirac spinor $G_{n\kappa}(z)$ will not be finite everywhere unless

$$a = 1 + \delta + \lambda - \chi = -n, \quad n = 0, 1, 2, 3, \ldots \tag{28}$$

from which we have

$$\lambda = -\frac{(n+1)^2 + \overline{\mathcal{A}} + \kappa(\kappa-1) + (2n+1)\delta}{2(n+\delta+1)}. \tag{29}$$

Using (7) and (12) into (17), we have an explicit expression for the energy eigenvalues of the Dirac equation with the Manning-Rosen potential

$$M^2 - E_{n\kappa}^2 + C(M + E_{n\kappa}) = \frac{1}{b^2}\left[\frac{(n+1)^2 + \overline{\mathcal{A}} + \kappa(\kappa - 1) + (2n+1)\delta}{2(n+\delta+1)}\right]^2. \tag{30}$$

This eigenvalue equation is still invariant under $\alpha \leftrightarrow 1-\alpha$. It can be seen from (30) that the pseudospin symmetric limit leads to quadratic energy eigenvalues. Hence, the solution of this equation consists of positive and negative energy eigenvalues for each $n$ and $\kappa$. In the pseudospin limit, it has been shown that there are only negative energy eigenvalues and no bound positive energy eigenvalues exist [7].

By using (27) we can write the lower spinor component

$$G_{n\kappa}(z) = N(1-z)^{1+\delta} z^{\lambda} {}_2F_1(-n, n+2+2(\delta+\lambda), 2\lambda+1; z) \tag{31}$$

where $N$ is the normalization constant. Moreover, it can be easily seen from (19) and (31) that when $\delta > 0$ and $\lambda > 0$, the upper spinor component $F_{n\kappa}(z)$ and lower spinor component $G_{n\kappa}(z)$ are satisfy the boundary conditions for the bound-state.

## 3 Discussion

In this section, we are going to study two spacial cases of the energy eigenvalues given by (18) and (30). First, let us study $l = 0$ ($\kappa = -1$) and $\tilde{l} = 0$ ($\kappa = 1$) case. It is shown from (14), (18), (26) and (30) that

$$M^2 - E_{n,-1}^2 - C(M - E_{n,-1}) = \frac{1}{b^2}\left[\frac{(n+1)^2 - \mathcal{A} + (2n+1)\delta_+}{2(n+\delta_+ + 1)}\right]^2 \tag{32}$$

and

$$M^2 - E_{n1}^2 + C(M + E_{n1}) = \frac{1}{b^2}\left[\frac{(n+1)^2 + \overline{\mathcal{A}} + (2n+1)\delta_-}{2(n+\delta_- + 1)}\right]^2 \tag{33}$$

where $\delta_\pm = \frac{1}{2}(-1 + \sqrt{1 \pm 4\xi_\pm})$. If one set $C = 0$ into (33), the result is related to (33) in [26].

Second, when we set $\alpha = 0$ or 1, the Manning-Rosen potential reduces to the Hulthén potential. Moreover, if taking $b = 1/a$ and defining $A/\Lambda b^2 = Ze^2 a$, we can obtain

$$M^2 - E_{n\kappa}^2 - C(M - E_{n\kappa}) = \left[Ze^2 \frac{(M + E_{n\kappa} - C)}{2(n+\kappa+1)} - \frac{(n+\kappa+1)}{2}a\right]^2 \tag{34}$$

and

$$M^2 - E_{n\kappa}^2 + C(M + E_{n\kappa}) = \left[Ze^2 \frac{(M - E_{n\kappa} + C)}{2(n+\kappa)} + \frac{(n+\kappa)}{2}a\right]^2. \tag{35}$$

These results coincide with that of reference [14].

## 4 Conclusions

In this study, we have presented bound-state solutions of the Dirac equation for the Manning-Rosen potential by applying an approximation to the spin-orbit coupling term. We have obtain an explicit expiration to the energy eigenvalues for arbitrary $\kappa$ state under condition of the spin symmetry and pseudospin symmetry concept. Approximate two Dirac spinors are expressed in terms of generalized hypergeometric functions. Moreover, we find that the energy eigenvalues remain invariant by mapping $\alpha \leftrightarrow 1-\alpha$. Finally, two special cases $\kappa = \mp 1$ and $\alpha = 0$ or 1 have been studied. We have seen that second special case reduces to the Hulthén potential and results are related to reference [14].

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